Update on some accuracy and uncertainty topics in DEMIX

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Abstract— **After the Geomorphometry 2023 meeting where the draft DEMIX report was discussed we have prosecuted the relevant research in four directions. First one is to establish an analytical relationship between DEM accuracy and cell size. We were able to show that this is possible, but it depends on the selected interpolation rule. Such choice is not intrinsic to the DEM as a dataset, and thus it is outside the DEM producer control. Second one is to do something similar for partial derivatives, leading to slope accuracy constrains in terms of elevation accuracy and cell size. Third one is regarding roughness, and the findings will be presented in Geomorphometry 2025. The relationship between early operational definitions of roughness within DEMIX and partial derivatives of the topographical surface was established. In general it is a function of the first and second order partial derivatives, but we show that with little effort they might be tied to just second order ones. The fourth one will presented also in Geomorphometry 2025. It describes ongoing work about the direct use of LiDAR datasets to estimate partial derivatives of the surface and its uncertainty at irregularly located points. We will show that we can go beyond, allowing to control the uncertainty by automatically changing the slope computation algorithm.**

I. INTRODUCTION

The DEMIX initiative [1] conducted a comparison exercise between global DEM of 1 arc second resolution. The task was organized in three sub groups (SG1, SG2 and SG3). This communication describes some research carried out after the end of activities of SG2 reported at the JRC Technical Report [2].

II. ERROR BUDGET FOR ELEVATION

Modern DEM offer higher resolution (smaller cell size) and higher accuracy as well. However, despite reasonable to have one, no closed form relationship links both. Early attempts only presented empirical proposals, certainly validated by experiments. In [3] we established analytically such relationship for the traditional elevation accuracy RMSE metric, showing some limitations of the empirical proposals. We were able to prove that the accuracy as MSE is bounded by the sum of a constant term and some monomial of the cell size h. Given the interpolation method the power exponent is a constant and the monomial coefficients dependent on certain terrain characteristics. The coefficients are computable. As a practical application, to produce a DEM with prescribed accuracy the operator can either decide the cell size for a given instrument, or tune the instrument accuracy for a given cell size.

To ease the analysis we choose to use as the preferred quantity the squared RMSE (thus, MSE hereinafter). After some detailed analysis involving Taylor expansions around control points the general relationship found is

$$MSE_{\text{model}} \leq MSE_{\text{data}} + C_1(\mathbf{e})h^p + C_2 h^{2p} \qquad (1)$$

We use the model suffix to denote the union of the raw data in raster format (the DEM) together with the chosen interpolation algorithm. The left hand side is the intended squared accuracy of the DEM+interpolation algorithm. It is bounded by the sum of three non-negative terms. The first one is the MSE of the grid elevation values themselves. It can only be computed when the control points are exactly located over the grid points. Notice that, provided the elevations are compared to reference ones exactly located at the same places, there is no interpolation degradation. Let's collect the grid value differences to a reference in a vector named **e**. If the elevation values are perfect, the vector norm is zero. Back to Eq. 1: its second term vanishes when the norm of **e** is zero. Otherwise, the second term evolves as certain power p of the cell size h. A third term, which does not vanish for perfect elevation data, evolves as a power of h with exponent 2p.

The value of p is equal to 1 for the case of Nearest Neighbor interpolation, and is equal to 2 for the Bilinear interpolation. If, as in DEMIX, all control points are located exactly on grid nodes, no interpolation will be required and the bound is simply the first term. Otherwise the three term model should be applied. Coefficients C_2 can be estimated from available data, because it is a function of the partial derivatives of the topographic surface. On the contrary, coefficient $C_1(\mathbf{e})$ requires not only partial derivative estimates but also knowing the elevation error at selected grid nodes. They are usually not available so such term is only retained for theoretical reasons.

Other interpolation methods (Radial Base Functions, Kriging, Inverse Distance Weighting, etc.) were not considered because they are not exact for polynomials surfaces.

The case of DEMIX is too special, because no interpolation algorithm is involved. It should be desirable that DEM producers report precisely such first term. Things change when the control points are not located precisely over grid points. The practical consequence of Eq. 1 is to help restrict the value of h in order to attain a prescribed accuracy. Even with perfect elevation values, if the cell size of the DEM is too large its accuracy will be also large. We are here concerned to confirm the accuracy requirements for a DEM to be considered as a reference, which (according to Eq. 1) also poses restrictions on its cell size. To be a reference dataset their accuracy as RMSE must satisfy

$$RMSE_{ref} \le RMSE_{cand}/3 \quad (2)$$

We use the suffix ref for reference and cand for candidate. After squaring both terms we find a relationship between their MSE, with a 1/9 factor. It can be assumed that the unknown coefficient C_2 stands the same both for reference and candidate DEM. On the contrary, it is not so clear the situation with $C_1(\mathbf{e})$. Under the assumption that it is similar we can try to verify that the 1/9 factor holds for each of three terms.

Eq. 1 applies also to the reference dataset. Its reported accuracy should have been computed w.r.t. to independent reference data, which for example might be located anywhere in the domain. Thus, from Eq. 1 we know that its dataset accuracy is bounded from above by the reported accuracy. In a worst case scenario, we can assume that such reference dataset accuracy is smaller than the candidate dataset accuracy for a factor of at least 1/3, so the first term is smaller by a factor of 1/9. The second term is linear in h. Assuming that the coefficient $C_1(\mathbf{e})$ is similar, we can request that the reference h should be smaller to the candidate h by a factor of 1/9 in the case of Nearest Neighbor interpolant (p=1), but it will suffices to be 1/3 if we use the Bilinear interpolant (p=2). The requirement for h due to the third term will be automatically satisfied.

Thus, in the general case where the interpolant to be used is unknown, the worst situation requires that the reference dataset has a resolution h smaller by 1/9 to the one of the candidate set. The requirement could be relaxed by using a more sophisticated interpolant. In the particular case of DEMIX the interpolant does not play a role provided both the grid of the reference dataset overlap with the candidate one, and the elevations themselves are accurate enough.

III. ERROR BUDGET FOR PARTIAL DERIVATIVES

Within DEMIX the accuracy has been computed using as reference values those available from higher resolution, higher accuracy DEMs exactly at the grid points. It was assumed that the partial derivatives computed from there are automatically suitable reference values. However, there were reasons to cast doubts so a formal development was carried out. The conclusion is the following expression, which relates the MSE of the partial derivative in terms of the MSE of the elevation data and the cell size

$$MSE_{derivative} \le C_0 * MSE_{data} + C_1 * \sqrt{MSE_{data}} * h^p + C_2 * h^{2p} \quad (3)$$

As before, there are three non-negative terms. The first one is only dependent on the elevation accuracy of the reference dataset, and is independent of the cell size. The second one vanishes either when the cell size goes down to zero or the elevation dataset is perfect. The third one is independent on the elevation accuracy, and is affected by a known power of the cell size. The coefficient is not readily computable, because it is affected by both the shape of the topographical surface as well as the error surface. If the partial derivatives are estimated with the Evans-Young formulae [4] then the value of p is 2.

Given that the bound is valid either for the reference dataset or the candidate dataset, it is possible to check its mutual relationship. As before, the relationship between their MSE should have a 1/9 factor. Assuming that the unknown coefficients C_0, C_1 and C_2 stands the same both for reference and candidate dataset we can try to verify that the 1/9 factor holds for each three terms.

The relationship for the first term is trivially satisfied. For the second we should request that

$$C_1 * \sqrt{MSE_{ref}} * h_{ref}^p \le C_1 * \sqrt{MSE_{cand}} * h_{cand}^p / 9 \quad (4)$$

which leads to

$$\sqrt{\frac{MSE_{ref}}{MSE_{cand}}} * \left(\frac{h_{ref}}{h_{cand}}\right)^p \le \frac{1}{9} \quad (5)$$

First factor is less than 1/3, so it will be enough if the second one is also less than 1/3

$$\left(\frac{h_{ref}}{h_{cand}}\right)^p \leq \frac{1}{3} \qquad (6)$$

Thus, for p equal to 2 it will be enough that h_{ref} is less than half the h_{cand}. With such a request, the third term automatically satisfies the 1/9 relationship, thus establishing a feasible requirement for the derivatives computed from the higher resolution, higher accuracy reference dataset.

The abovementioned requirement ignores the scale effect, which manifest itself because the derivative estimates using very different cell size h are not readily comparable. This is a well known problem and we will not address it here. The problem are still under analysis, so no final conclusion have yet been offered.

IV. ESTABLISHING A RELATIONSHIP BETWEEN PROPERTIES OF THE TOPOGRAPHIC SURFACE AND ROUGHNESS AS COMPUTED

Within DEMIX it was proposed from the inception that accuracy criteria will involve not only the traditional elevation one, but also slope and roughness. Early operational definition of roughness involves the standard deviation of slope estimates taken over a 3x3 window [5]. Using criteria like the Evans-Young [4] this involves in practice a 5x5 window. As presented, this was an algorithmic definition, explaining how to compute it. But it is not a formal definition, relating the attained value to other properties of the topographic surface. Despite this might not be a real concern for elevation data defined over a regular grid, it left undefined how to compute the same roughness using other elevation dataset formats like TIN, contour lines, or even from sparse elevation values coming from LiDAR or field survey. This is different from other magnitudes, like slope, which can be computed and can be comparable in any dataset formats

The final DEMIX choice was not to use roughness computed over a 5x5 window but over a 7x7 one, but anyway we will present here our results. After a cumbersome Taylor expansion of the elevation around the central point and taking the limit when the cell size goes down to zero we were able to prove that the roughness defined as the standard deviation of the slope estimated over 9 points of a 3x3 window is a function of both first and second partial derivatives taken at the central point. Unfortunately, its expression changes if the slope at the central point is zero or not. Since this is cumbersome, we considered other options. One alternative for the roughness definition were also informally considered during the task. It proposes using the standard deviation of the slope computed over a detrended elevation dataset. The trend was estimated using a tangent plane going through the central point. With such modification, we were able to prove that the roughness is now independent of the first order derivative at the central point, and remains a function of just the second derivative ones.

The suggested approximation for the roughness using a finite h is only first order accurate. However, once we have defined it in terms of the second order partial derivatives we can resort to other options. Florinsky [6] proposed estimates for higher order derivatives of the topographical surface using a 5x5 windows. In [7] it has been proved that the estimates are fourth order accurate w.r.t. the cell size. If they are inserted in the formal definition of roughness the new estimate becomes also higher-than-one order accurate. That opens the door to produce an uncertainty estimate as the absolute difference between the given operational value and the last one suggested. In turn, that paves the way to develop a criteria to accept/deny the reference roughness value coming from a higher resolution, higher accuracy DEM, provided that scale effects are neglected. I envision a workaround to properly cope with the scale effects, but this has yet not fully developed.

V. BEYOND ESTIMATE: A PATH TO NOT JUST ESTIMATE BUT TO CONTROL SLOPE UNCERTAINTY

In [7], and also motivated by DEMIX activities, it has been presented a procedure able to assess the contribution of finite value h cell size to the uncertainty of the partial derivatives estimates. In general, the uncertainty is bounded by the absolute value of the difference between numbers computed from high and low order formulae. The order is a property of the formulae, and in the paper a number of popular algorithms were analyzed. It was concluded that most of them are of second order, and among the set only the one by Jones [8] is of first order while the one due to Florinsky [6] is of fourth order. The conclusions of [7] are only valid for elevations defined over a regular grid, since all the methods considered are tied to such data organization.

While considering an extension of such analysis to elevation data coming from LiDAR it becomes apparent that something else can be done. The activity is part of an ongoing project, supported by the Spanish government. The goal of [7] was just to estimate the uncertainty, given that the partial derivative themselves are available. For LiDAR there are not such formulae, so we need to start from the beginning. To address the issue we established a new connection to the partial differential equations (PDE) literature. Despite developed independently, all the traditional formulae like Evans-Young [4], Horn [9], Jones [8], Florinsky [6], etc. are indeed particular cases of the Finite Difference approach. In the PDE literature, a first step to find a solution is to declare as unknown the function value at every grid point. As a second step the partial derivatives are estimated in terms of those unknowns. After that, what is known as the Strong Form Approach requires that the PDE itself is then imposed to the partial derivatives producing a nonlinear equation to be satisfied at every cell point. The solution arises after finding function values that satisfy such equation, at least approximately. In the Numerical Analysis community the procedure is deemed incomplete unless an error/uncertainty estimate is offered, so procedures like the ones presented in [7] are essential to satisfy such need. If the value of the uncertainty estimate is above a prescribed tolerance, the cell size is diminished and the procedure repeated until necessary. Thus, the concept of control of the committed uncertainty is central to the whole procedure. As it is a deterministic context, the PDE literature usually use the name error for what here we will denote as uncertainty.

Aside from the Finite Difference approach which operates over regular grids, other ones were developed. In particular, the so called Mesh-Free or Meshless methods. They again use as unknowns the function values defined over a cloud of points not regularly located. They support also the Strong Form Approach. Algorithms to estimate the partial derivatives are embedded in the codes, as well as its error/uncertainty estimates. Again, imposing the PDE creates algebraic nonlinear equations to be solved. Once solved, they also produce an uncertainty estimate. If the uncertainty is within tolerance, the solution is accepted. If not, something needs to be done. The novelty of these methods is that, unlike the Finite Difference ones, the cloud of points might not be easily expanded by adjusting just one parameter like the cell size. Instead, the point location might be defined a priori. The LiDAR placement of points is a good example of such situation. There is no possibility to add extra points. The alternate solution applied by the Meshless approach is to produce a more accurate partial derivative estimate by somewhat arbitrarily increasing its accuracy order. Notice that for most of the traditional methods mentioned for the regular grid the order was constant. In the Meshless approach the partial derivative estimation method order is increased as needed. This is possible for irregular located data values, and it is also possible for regular ones (see, for example, [10]). This assertion is valid for partial derivatives of any order. I am not aware why such alternative is not commonly used in geomorphometry even for regular grids, considering progressively second, fourth and higher order estimates until a prescribed low uncertainty is achieved.

VI. CONCLUSIONS

- We have developed an analytical relationship between DEM accuracy and the cell size. The relationship explicitly requires considering the interpolation method, a fact not strictly tied to the data. This has implications for data producers as well as data users. In the DEMIX context we are able to specify requirements for DEMs intended to operate as a reference.
- We have developed an analytical relationship between the partial derivative accuracy and both the elevation accuracy and the cell size.
- We have found the analytical, exact expression, relating the two different definitions of roughness to local partial derivatives. It has been shown that the limit value of the roughness for the second definition only involve second order derivatives.
- Future works might develop more refined uncertainty estimates which properly deal with scale issues.

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