A Differential Equation for Specific Catchment Area

J. C. Gallant[1], M. F. Hutchinson[2]

[1]CSIRO Land and Water, GPO Box 1666, Canberra ACT 2601, Australia
Telephone: +61 2 62465734
Fax: +61 2 62465800
Email: John.Gallant@csiro.au

[2]Fenner School of Environment and Society, Australian National University, Canberra ACT 0200, Australia
Telephone: +61 2 61254783
Fax: +61 2 61250757
Email: Michael.Hutchinson@anu.edu.au

1. Introduction

Specific catchment area is one of the key land-surface parameters used in the fields of hydrology, geomorphology, pedology and ecology and many methods have been devised to estimate it from grid DLSMs. An accurate reference is required to test these methods, but specific catchment area has only been determined analytically for simple surfaces such as inclined planes and cones. To assess the results on the complex surfaces of natural terrain developers and users have resorted to comparisons between the results from different methods and visual inspection of the patterns of estimated specific catchment area.

This paper presents a differential equation that describes the rate of change of specific catchment area along a flow line. The equation can be solved numerically along a flow line that is derived numerically from a grid digital elevation model, allowing precise values of specific catchment area to be obtained at any location on a complex terrain surface.

The development of the concepts and exploration of the results and implications are addressed in more detail in Gallant and Hutchinson, (in preparation).

2. Stream Tubes and Specific Catchment Area

Specific catchment area, contributing area (or total catchment area, TCA) and flow width are all defined in terms of a *stream tube* (Onstad and Brakensiek, 1968) consisting of two adjacent flow lines terminating at the downstream end on the opposite ends of a contour segment (Fig. 1). Specific catchment area *a* is defined as

$$a = \lim_{w \to 0} \frac{A}{w} \tag{1}$$

where A is contributing area, the area of the stream tube, and w is the width of the lower end of the stream tube.

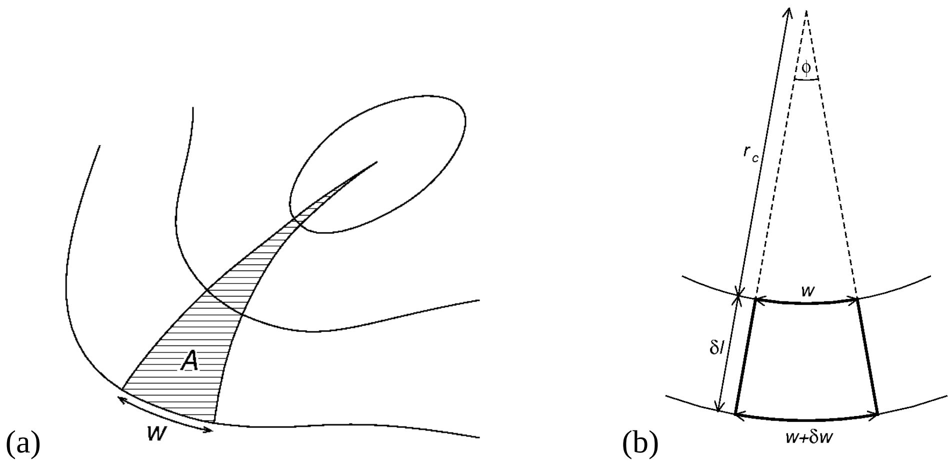

Figure 1. (a) A stream tube of area *A* defined by a contour segment of length *w*.
(b) A segment of stream tube showing change in contour length determined by contour line curvature.

The area *A* at distance *l* from the hilltop is the integral of flow width *w* with respect to flow length:

$$A(l) = \int_0^l w(u)du \tag{2}$$

or equivalently

$$\frac{dA}{dl} = w \tag{3}$$

Consideration of the geometry of a short segment of stream tube where the radius of curvature of the contour lines is r_c leads to an expression for the rate of change of contour segment width *w*:

$$\frac{dw}{dl} = \frac{w}{r_c} = wK_c \tag{4}$$

where $K_c = \frac{1}{r_c}$ is the plan curvature (or contour curvature), defined here to be positive for divergent regions (as shown in Fig. 1(b)) and negative for convergent regions.

The equations for $\frac{dA}{dl}$ and $\frac{dw}{dl}$ allow a solution to the variation of specific catchment area *a* along a flow line starting from the definition of *a* at a point:

$$a = \lim_{w \to 0} \frac{A}{w} \tag{5}$$

$$\frac{da}{dl} = \frac{d}{dl} \lim_{w \to 0} \frac{A}{w} \tag{6}$$

$$= \lim_{w \to 0} \frac{w\dfrac{dA}{dl} - A\dfrac{dw}{dl}}{w^2} \tag{7}$$

$$= \lim_{w \to 0} \frac{w^2 - AwK_c}{w^2} \tag{8}$$

$$= \lim_{w \to 0} \left(1 - \frac{AK_c}{w} \right) \tag{9}$$

$$= 1 - K_c \lim_{w \to 0} \frac{A}{w} \tag{10}$$

$$\frac{da}{dl} = 1 - K_c a \tag{11}$$

Equation 11 describes the rate of change of specific catchment area *a* along the flow path. It is a non-linear differential equation that can be integrated numerically along a flow path constructed starting from a hilltop with $a = 0$. Under suitable conditions it can be solved analytically, although most of the analytical results can be obtained more easily by solution from first principles.

This equation has some interesting properties. The two terms on the right hand side relate to the two sources of change in specific catchment area: the constant term represents the increment due to increasing length of the flow line, while the second term captures the effects of convergence and divergence. In divergent terrain ($K_c > 0$) the two terms compete and $\frac{da}{dl}$ may increase or decrease along the flow path depending on the size of *a* and K_c. If $a = \frac{1}{K_c}$ the two terms balance exactly and specific catchment area remains constant along the flow path. In convergent terrain the two terms are both positive leading to an exponential increase in specific catchment area.

The use of a single flow line to determine specific catchment area becomes untenable in strongly convergent areas and in channels, where total catchment area is a more relevant quantity, but the results of Equation 11 are applicable over most of the landscape. A robust method to determine where Equation 11 ceases to be applicable; the condition $a > 5l$ is suggested in Gallant and Hutchinson (in preparation).

3. Numerical Solution

The solution of (11) on a grid DLSM requires:

- an interpolation method to create a smooth surface so that first and second derivatives can be computed at any point
- a method to construct flow lines on that surface
- a method to integrate (11) along a flow line

We use a biquadratic interpolation method (de Boor, 1978) to provide a continuous surface with continuous first derivatives. The flow lines are constructed as short straight line segments using a midpoint method with adaptive step size. At the start of each step, the local flow direction (aspect) is determined using the first derivatives of the surface, a trial half-step is taken, the direction is re-computed at the half-step point then a full step is taken in that new direction. The step size is adjusted so that the directions at the beginning and end of each step are quite close (dot product of unit vectors is greater than 0.99).

Integration of (11) is achieved by analytical solution for each segment of the flow line with appropriate assumptions on the variation of K_c along the segment. In most cases the form:

$$K_c(l) = \frac{1}{c_1 l + c_0} \tag{12}$$

is used, yielding the solution:

$$a(l) = \frac{c_0 + c_1 l}{1 + c_1} + \left(a_0 - \frac{c_0}{1 + c_1}\right)\left(\frac{c_0 + c_1 l}{c_0}\right)^{-1/c_1} \tag{13}$$

provided $c_1 \neq -1$ and $c_1 \neq 0$ and that K_c does not change sign along the line segment. If any of those conditions are not satisfied, an alternate solution with constant K_c is used:

$$a(l) = \frac{1}{K_c} - \left(\frac{1}{K_c} - a_0\right)e^{-K_c l} \tag{14}$$

The integration commences at the top of the stream line with $a = 0$ and proceeds segment by segment to the end of the line.

The accuracy of the numerical solution will depend on the accuracy with which the flow line is constructed and the accuracy of the integration of Equation 11 along the line. The choice of the interpolation method will also have an impact on the results, although this is more a matter of choice than accuracy. In our experiments, we have used conservative parameters for the construction of the line so that many short line segments are constructed and further refinement of the line does not produce noticeable changes to either the path of the line or the integration of Equation 11. Gallant and Hutchinson (in preparation) demonstrate that the numerical results are indistinguishable from full analytical solutions on surfaces where analytical solutions are available (planes and cones).

4. Comparisons with Conventional Methods

The most valuable application of Equation 11 is as a reference against which the approximate (and much more efficient) methods of calculating specific catchment area can be compared. Figure 2 shows a flow path and stream tube calculated from a 20 m resolution DLSM in the Brindabella Ranges near Canberra in southeastern Australia (35° 21′S 148° 49′E). Figure 3 shows specific catchment area calculated along the central flow line using Equation 11 along with estimates from the most commonly used conventional methods: D8 (O'Callaghan and Mark, 1984); slope-weighted multiple flow direction here referred to as M8 (Freeman, 1991; Quinn *et al.*, 1991, 1995); DEMON (Costa-Cabral and Burges, 1994); and D∞ (Tarboton, 1997).

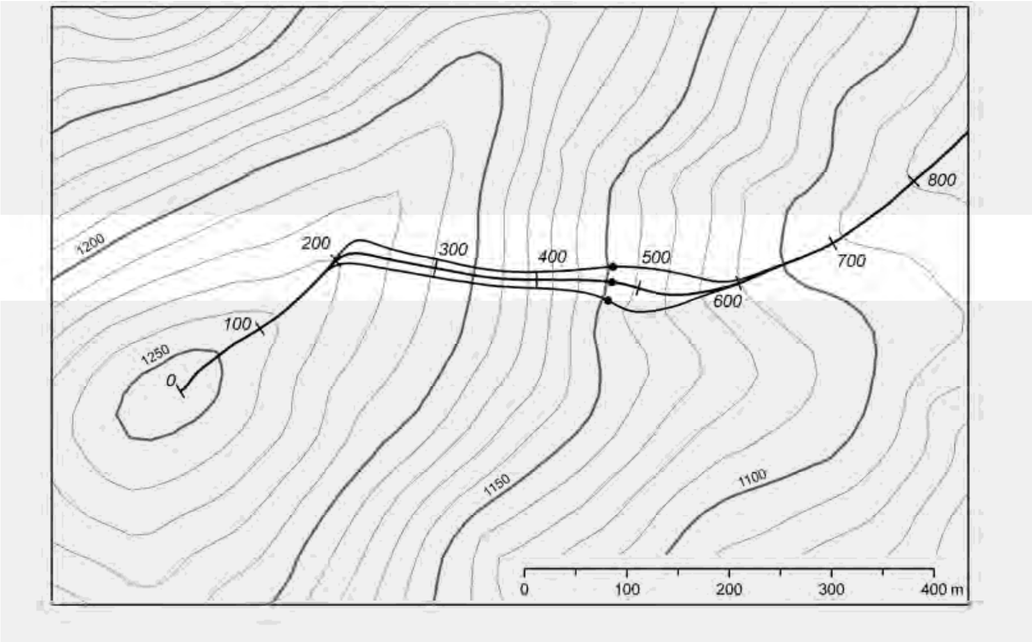

Figure 2. Flow lines derived from a 20 m resolution DLSM

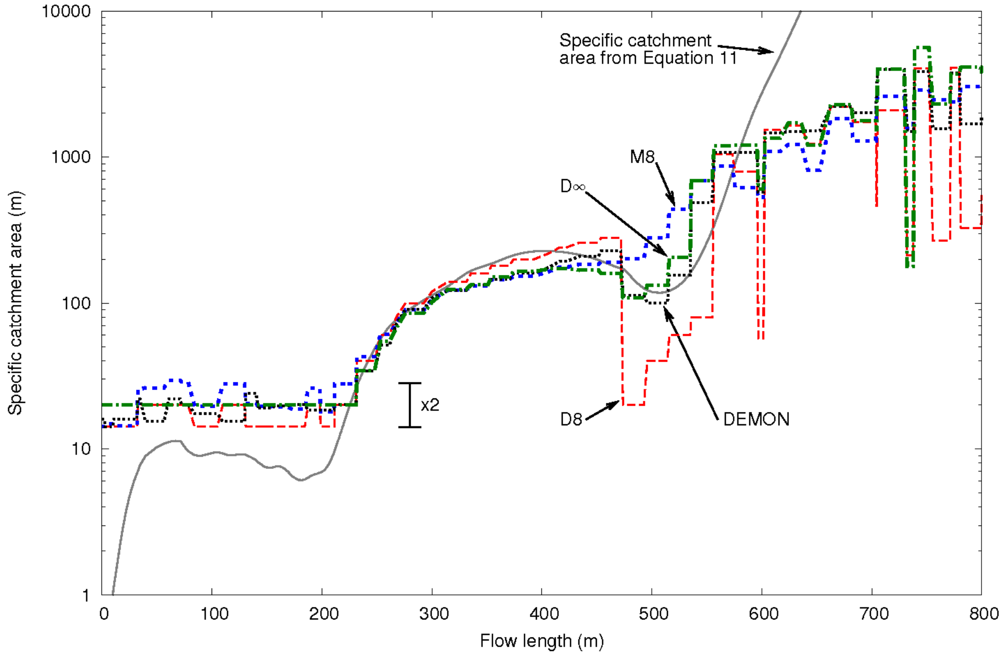

Figure 3. Specific catchment area along the flow line of Figure 2 computed using Equation 11 and four conventional methods.

Along the divergent ridgeline (flow length of 0 – 200 m) all the conventional methods over-estimate specific catchment area by about a factor of two. This is largely due to the way in which flow width is estimated in these methods; flow widths range from 1 to 1.414 times the cell size, whereas on a ridgeline flow leaves the cell over much of the perimeter of the cell.

Along the planar hillslope (200 – 400 m) all methods perform quite well with D8 giving the closest results to Equation 11. This result reflects the orientation of the flow path in a cardinal direction, where the known deficiencies of the D8 method do not occur.

Between 400 and 600 m the terrain is gently divergent then convergent. The D8 method is unable to capture this subtle variation, M8 performs better but still poorly while DEMON and D∞ behave quite well. D∞ appears to best capture the decline in specific catchment area around 450 m.

In the more convergent area beyond 600 m the result from Equation 11 increases exponentially, which is an unrealistic result that can be traced back to the assumption in Equation 1 that flow always responds to local curvature.

5. Conclusions

Equation 11 provides a method for accurately calculating specific catchment area at a specified point on a grid DEM without having to separately calculate catchment area and flow width. This method provides for the first time a means of testing approximate grid-based methods in complex terrain. A brief comparison with existing methods shows that the D∞ method best captures the variations in specific catchment area along the single flow line examined, although all methods over-estimate specific catchment area on divergent ridge lines.

References

Costa-Cabral M and Burges S, 1994, Digital Elevation Model Networks (DEMON): A model of flow over hillslopes for computation of contributing and dispersal areas. *Water Resources Research* 30(6):1681-1692.

de Boor C, 1978, A practical guide to splines. Springer-Verlag, Berlin.

Freeman T, 1991, Calculating catchment area with divergent flow based on a regular grid. *Computers and Geosciences* 17(3):413-422.

Gallant J and Hutchinson M, in preparation. Exact calculation of specific catchment area on grid DEMs, *to be submitted to Water Resources Research.*

O'Callaghan J and Mark D, 1984, The extraction of drainage networks from digital elevation data. *Computer Vision, Graphics and Image Processing* 28(3):323-344.

Onstad C and Brakensiek D, 1968, Watershed simulation by the stream path analogy. *Water Resources Research* 4(5):965-971.

Quinn P, Beven K, Chevallier P and Planchon O, 1991, The Prediction of Hillslope Flow Paths for Distributed Hydrologic Modelling using Digital Terrain Models. *Hydrological Processes* 5(1):59-79.

Quinn P, Beven K and Lamb R, 1995, The $\ln(a/\tan\beta)$ index: How to calculate it and how to use it within the TOPMODEL framework. *Hydrological Processes* 9(2):161-182.

Tarboton D, 1997. A new method for the determination of flow directions and upslope areas in grid digital elevation models. *Water Resources Research* 33(2):309-319.