Monkey, Starfish and Octopus Saddles

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***Abstract*—This paper provides expressions and results for a family of saddle surfaces that includes the *simple saddle* and lesser-known *monkey saddle* as well as an infinite sequence of higher-order saddles that includes what could be called *starfish* and *octopus saddles*. Saddles often occur in real topography along drainage divides. Traversing a drainage divide involves repeatedly moving from a peak down a descending ridgeline to a saddle point and then up an ascending ridgeline to another peak. Peaks that are similar to monkey saddles also occur in real topography and are discussed. A derivation is given that provides polynomial expressions for an infinite family of high-order saddle surfaces. In addition, interesting general expressions are given for the plan, profile and streamline curvatures of these surfaces. While interesting on their own, these surfaces can also be used as test surfaces for geomorphometric analysis and algorithms.**

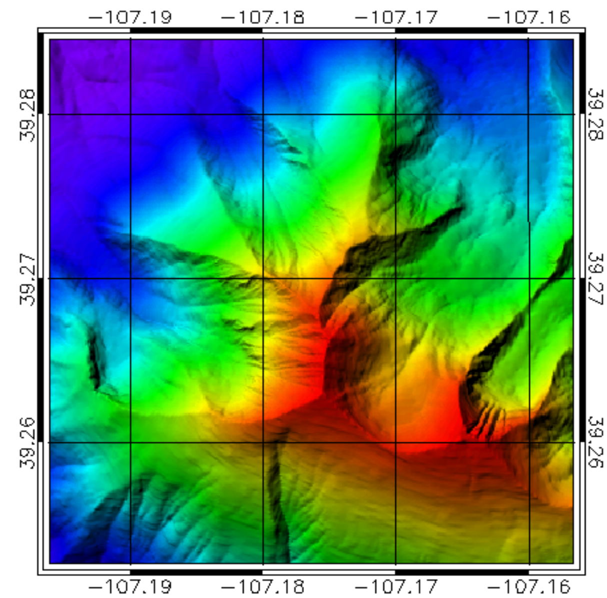

Fig. 1. Shaded relief image for Mount Sopris, Colorado.

I. Introduction

A *simple saddle* surface, with the saddle point located at the origin, is given by

$$f(x,y) = x^2 - y^2. \quad (1)$$

This function is also known as a *hyperbolic paraboloid*. The function $f(x,y) = 2\,x\,y$ is a rotated version of the same surface. A more exotic type of saddle surface is the *monkey saddle*

$$f(x,y) = x^3 - 3\,x\,y^2. \quad (2)$$

This saddle is so-named because it could be used by a monkey; it has places for two legs and a tail. Figure 1 shows a color shaded relief image for Mount Sopris, in Colorado, USA. Mount Sopris has two peaks of equal elevation known as West Sopris and East Sopris. West Sopris, though similar to a monkey saddle, is actually a peak and not a true saddle. A simple saddle occurs between the two peaks.

II. Higher-Order Saddle Surfaces

Polynomial expressions for a whole family of higher-order saddle surfaces can be obtained as follows. Thinking in terms of polar coordinates, we see that the simple saddle has 2 minima and maxima as we move along a circle centered at the origin while the monkey saddle has 3 of each. Since $\cos(n\,\theta)$ has n minima and n maxima as θ varies from 0 to 2π, any function in polar coordinates (r,θ) of the form $f(r,\theta) = F(r)\cos\left[n\left(\theta+\theta_0\right)\right]$ corresponds to a saddle of order n. We can generate a particular family of n-saddles such that the first two are the simple saddle and monkey saddle by taking $F(r) = r^n$ and $\theta_0 = 0$ to get

$$f_n(r,\theta) = r^n \cos\left(n\,\theta\right). \quad (3)$$

Converting directly to Cartesian coordinates we get

$$f_n(x,y) = \left(x^2+y^2\right)^{n/2} \cos\left[n\left(\arctan(y/x)\right)\right] \quad (4)$$

but unlike (1) and (2), this isn't in the form of a simple polynomial. To get an expression in that form, consider the complex function

$$z(r,\theta) = r^n e^{i\,n\,\theta} = r^n\left[\cos(n\,\theta) + i\,\sin(n\,\theta)\right] \quad (5)$$

where i is the imaginary number given by $i=\sqrt{-1}$. The second equality is from *Euler's identity* and shows that the real part of $z(r,\theta)$ is our n-saddle function, $f_n(r,\theta)$. (Let $g_n(r,\theta)$ denote the imaginary part, for later use.) Now since $e^{i\,n\,\theta} = \left(e^{i\,\theta}\right)^n$, we also have

$$\begin{aligned} z(r,\theta) &= r^n\left[\cos(\theta) + i\,\sin(\theta)\right]^n && (6)\\ &= \left[r\cos(\theta) + i\,r\sin(\theta)\right]^n && (7)\\ &= (x + i\,y)^n && (8)\\ &= \sum_{k=0}^{n}\binom{n}{k} x^{n-k}\,y^k\,i^k. && (9)\end{aligned}$$

Here we've converted back to Cartesian coordinates and then applied the *binomial formula*. So the real part of this polynomial gives another expression for our family of n-saddles. Since imaginary terms only occur for odd values of k, we get the real part by restricting the sum to even values of k

$$f_n(x,y) = \sum_{\substack{k=0,\\ k\ even}}^{n}\binom{n}{k} x^{n-k}\,y^k\,i^k. \quad (10)$$

Keep in mind that $i^2=-1$, $i^3=-i$ and $i^4=1$. From this formula, $f_1(x,y)=x$ and the first 4 saddle surfaces are found to be

$$\begin{aligned} f_2(x,y) &= x^2 - y^2 && (11)\\ f_3(x,y) &= x^3 - 3\,x\,y^2 && (12)\\ f_4(x,y) &= x^4 - 6\,x^2\,y^2 + y^4 && (13)\\ f_5(x,y) &= x^5 - 10\,x^3\,y^2 + 5\,x\,y^4. && (14)\end{aligned}$$

It seems reasonable to call an order 5 saddle a *starfish saddle* (see Figure 4) and an order 8 saddle an *octopus saddle*. If we had taken the imaginary part of $z(r,\theta)$ instead

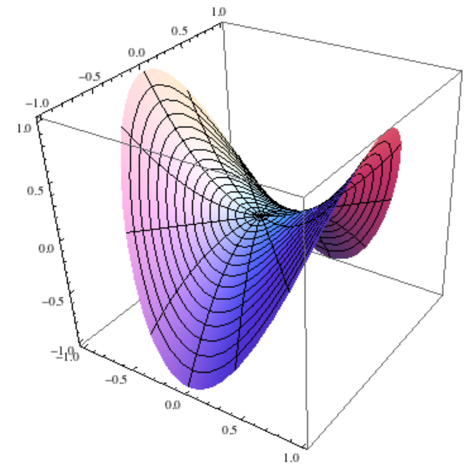

Fig. 2. A simple saddle, $f_2(x,y) = x^2 - y^2$.

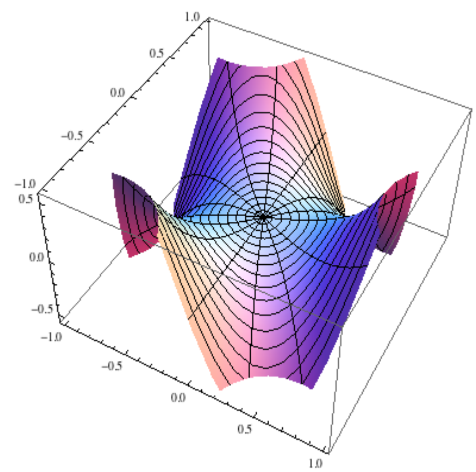

Fig. 3. A monkey saddle, $f_3(x,y) = x^3 - 3\,x\,y^2$.

of the real part, which we denoted earlier as $g_n(r,\theta)$, we would have obtained polynomial expressions for versions of our n-saddle family that have been rotated by 90 degrees. Note that $g_1(x,y)=y$ and the first 4 saddle surfaces are

$$\begin{aligned} g_2(x,y) &= 2\,x\,y && (15)\\ g_3(x,y) &= 3\,x^2\,y - y^3 && (16)\\ g_4(x,y) &= 4\left(x^3\,y - x\,y^3\right) && (17)\\ g_5(x,y) &= 5\,x^4\,y - 10\,x^2\,y^3 + y^5. && (18)\end{aligned}$$

A. Slopes of Saddle Surfaces

Slope, $S(x,y)$, can be defined for each point on a surface, $f(x,y)$, as the magnitude of the gradient vector, ∇f. In Cartesian coordinates we have $\nabla f = (f_x, f_y, 0) = f_x\,\hat{\imath} + f_y\,\hat{\jmath}$ and $S(x,y) = \sqrt{f_x^2 + f_y^2}$. The subscripts x and y here denote partial derivatives. In polar coordinates we have

$$\nabla f = (f_r, f_\theta/r) = f_r\,\hat{r} + (f_\theta/r)\,\hat{\theta} \quad (19)$$

$$S(r,\theta) = |\nabla f| = \sqrt{f_r^2 + (f_\theta/r)^2}. \quad (20)$$

Applying this formula for $S(r,\theta)$ to (3), we get

$$S(r,\theta) = n\,r^{n-1}. \quad (21)$$

This somewhat nonintuitive result shows that the slope for any of these saddle surfaces depends only on the distance from the origin, r. In mathematics, a *saddle point* is a point in the domain of a function where the function has slope zero (called a *stationary* or *critical point*) but that is not an extremum (a pit or peak). The origin ($r = 0$) is a saddle point for every saddle in our two saddle families.

B. Curvatures of Saddle Surfaces

In Cartesian coordinates, *plan curvature* of a surface, $f(x,y)$, is given by

$$\kappa_c(f) = -S^{-3}\left(f_y^2 f_{xx} - 2\,f_x\,f_y\,f_{xy} + f_x^2 f_{yy}\right). \quad (22)$$

We can use the *Mathematica* symbolic math software to quickly compute this for (4). Converting the result back to polar coordinates we get

$$\kappa_c(r,\theta) = \frac{(n-1)\,f_n(r,\theta)}{r^{n+1}}. \quad (23)$$

This is interesting because the resulting expression is so simple and just rescales the saddle surface itself by a function of r. *Profile curvature* is given by

$$\kappa_p = -S^{-2}\left(f_x^2\,f_{xx} + 2\,f_x\,f_y\,f_{xy} + f_y^2\,f_{yy}\right). \quad (24)$$

Computing this for (4) we get

$$\kappa_p(r,\theta) = \frac{-n(n-1)f_n(r,\theta)}{r^2}. \quad (25)$$

This is again simple and rescales the saddle surface itself by a function of r. *Streamline curvature* is given by

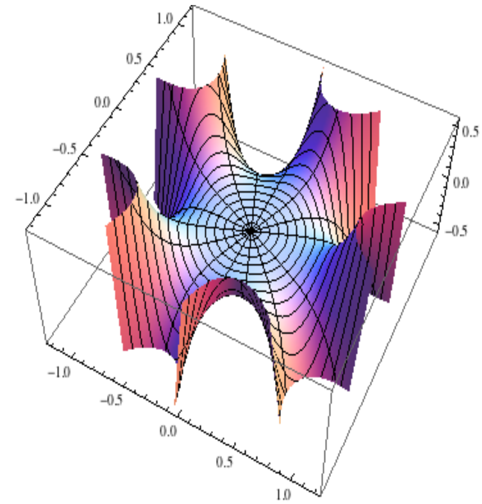

Fig. 4. A starfish saddle. $f_5(x,y) = x^5 - 10\,x^3\,y^2 + 5\,x\,y^4$.

$$\kappa_s = -S^{-3}\left[f_x\,f_y\,\left(f_{xx} - f_{yy}\right) + \left(f_y^2 - f_x^2\right)\,f_{xy}\right]. \quad (26)$$

See [1] for a discussion of plan, profile and streamline curvature. Computing this for (4) we get

$$\kappa_s(r,\theta) = \frac{(n-1)\,g_n(r,\theta)}{r^{n+1}}. \quad (27)$$

This is again simple and now rescales a rotated version of the saddle surface, as denoted previously by $g_n(r,\theta)$, by a function of r.

C. Classification of Saddle Surfaces

The *Gaussian curvature*, K, gives a measure of the overall or net curvature at a point (x,y) and can be computed for a surface, $f(x,y)$, as

$$K = \frac{f_{xx}\,f_{yy} - f_{xy}^2}{\left(1+S^2\right)^2}. \quad (28)$$

A point (x,y) is classified as *elliptic* if $K > 0$, *parabolic* if $K = 0$ and *hyperbolic* if $K < 0$. Peaks and pits are elliptic and the center point of a simple saddle is hyperbolic. Note also that peaks, pits and saddles are *critical points* where $S = 0$. Using (28), it can be shown that all points on a simple saddle ($n = 2$) are hyperbolic and all points except the origin are hyperbolic for higher-order saddles with $n > 2$. However, for the saddle surfaces with $n > 2$, the origin is a parabolic point. This means that the higher-order saddles are very flat in the neighborhood of the origin, but they are still saddle points.

III. A Wavy Skirt Surface

When we examine the West Sopris "monkey saddle" more closely, we find that the center point is actually a peak, not a saddle point. Linear transects that start at the center show that elevations *decrease* in a linear manner as we move outward along any of the 3 ridges or valleys. However, a slight generalization of a saddle surface which we will call a *wavy skirt surface* allows us to capture these observations, namely

$$f_n(r,\theta) = a\,r\,\left[\cos\left(n\,\theta\right) - b\right]. \tag{29}$$

As compared to a high-order saddle, this surface has linear r-dependence and a shifted function of θ. The origin is now a peak and no longer a saddle point. Fig. 5 shows an example where $n = 5$, $a = 1/2$, and $b = 3/2$. Note that the ridge tops of (29) occur where $\cos(n\,\theta) = 1$, or $\theta_k = 2k(\pi/n)$, $k \in \{0, ..., n-1\}$. Similarly, the valley bottoms occur where $\cos(n\,\theta) = -1$, or $\theta_k = (2k+1)(\pi/n)$. Elevation as a function of r is then given by $z_r(r) = a\,(1-b)\,r$ on a ridge top and $z_v(r) = -a(1+b)\,r$ in a valley bottom. So assuming $a > 0$, longitudinal profiles for both ridge tops and valley bottoms will be decreasing, linear functions of r only if $b > 1$. A surface given by (29) with $n = 3$ and $b > 1$ therefore provides a better model for a peak like West Sopris. However, the slope of this surface depends only on θ and is degenerate (i.e., it is multiple-valued) at the peak $(r = 0)$.

IV. Conclusions

Mathematical expressions for saddle surfaces in terms of polar coordinates often take the form of a product $f_n(r,\theta) = F(r)G(n\,\theta)$, where F is a function of r and and G is a periodic function of θ. Results from complex number theory can be used to derive equivalent, polynomial expressions for two high-order saddle families of this type in terms of Cartesian coordinates. The two families differ by a simple rotation. These families include

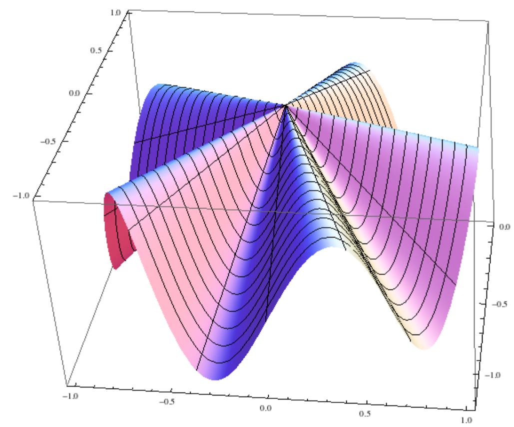

Fig. 5. A variant of the starfish saddle, here called a *wavy skirt* of order 5.

saddles of arbitrary order $n > 1$, and therefore include simple, monkey, starfish and octopus saddles as special cases. General expressions for slope and three types of curvature (plan, profile and streamline) are also given for these two saddle families. These results can be used as benchmark tests for the algorithms that are used in geomorphometry to analyze real topographic surfaces.

While real topographic surfaces have features that are very similar to monkey saddles and other high-order saddles, the author is unaware of cases that are true saddles with order $n > 2$. An examination of the west peak of Mount Sopris shows that elevations decrease linearly from the peak along any of the three ridges or three valleys that meet at the peak. A modified saddle surface called a *wavy skirt surface* was introduced in order to capture these features.

References

[1] Peckham, S.D. (2011) Profile, plan and streamline curvature: A simple derivation and applications (this issue).