

Mathematical Surfaces for which Specific and Total Contributing Area can be Computed: Testing Contributing Area Algorithms

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Abstract—In order to properly evaluate different algorithms for computing the total contributing area (TCA) and specific contributing area (SCA) from DEMs, it is important to have mathematical test surfaces for which these quantities can be computed in closed form. In previous work, the inverted cone and inclined plane have been used for this purpose because they are the only mathematical surfaces for which closed-form results were available. Note that the various algorithms differ most in terms of how well they work on divergent topographic surfaces where streamlines diverge, and several different algorithms based on multiple flow directions have been developed for this case. The purpose of this paper is to show how TCA and SCA can be computed mathematically and to provide several new examples of test surfaces for which results can be given in closed form. These new results are then briefly compared to results from an advanced method for computing TCA and SCA from DEMs called the *mass flux method*. More detailed comparisons to results from this and other methods will be presented later in a full paper.

I. INTRODUCTION

For a divergent, radially-symmetric surface, both the TCA and SCA can be computed in closed form (Gruber and Peckham, 2009). As shown in Figure 1, the TCA of a pixel with $\Delta y = \Delta x$, centered at $(x, y) = (i\Delta x, j\Delta x)$, where i and j are integers and $m = |i| + |j|$, is given by

$$A(i, j) = \begin{cases} \Delta x^2 (m + 1) / 2, & \text{if } i \neq 0 \text{ and } j \neq 0 \\ \Delta x^2 (m + \frac{3}{2}) / 2, & \text{if } i = 0 \text{ xor } j = 0 \\ \Delta x^2, & \text{if } i = 0 \text{ and } j = 0. \end{cases} \quad (1)$$

Even though this expression is exact, the middle case is responsible for an odd-looking “artifact” along the axes. The width of a pixel, as projected toward the origin, is

$$w(x, y) = \Delta x [|\sin(\theta)| + |\cos(\theta)|] \quad (2)$$

$$= \frac{\Delta x (|x| + |y|)}{\sqrt{x^2 + y^2}}, \quad (3)$$

where $\theta(x, y) = \tan^{-1}(y/x)$ and we used the identities $\sin(\tan^{-1}(x)) = x/\sqrt{1+x^2}$ and $\cos(\tan^{-1}(x)) = 1/\sqrt{1+x^2}$. The SCA is therefore given by

$$a(x, y) = \lim_{\Delta x \rightarrow 0} \frac{A(x, y)}{w(x, y)} = \frac{\sqrt{x^2 + y^2}}{2}. \quad (4)$$

It depends only on distance from the origin.

Note: If we instead place the origin/peak at a pixel corner, four pixels will share the peak and we get a single expression for all pixels, $A(i, j) = \Delta x^2 (m + 1) / 2$, where $m = |i + \frac{1}{2}| + |j + \frac{1}{2}|$, $x = (i + \frac{1}{2})\Delta x$, $y = (j + \frac{1}{2})\Delta x$. This change appears to remove the “artifact” associated with the middle case in equation (1).

II. BACKGROUND

A. Orthogonal Curvilinear Coordinate Systems

An orthogonal curvilinear (OC) coordinate system with coordinates u and v can be specified relative to Cartesian coordinates x and y by a transformation $x(u, v)$, $y(u, v)$. The inverse of the transformation is denoted by $u(x, y)$, $v(x, y)$. In order for curves of constant u and constant v to be orthogonal, we must have $x_u x_v + y_u y_v = 0$, where the subscripts denote partial derivatives with respect to u and v . In two dimensions, two functions of u and v

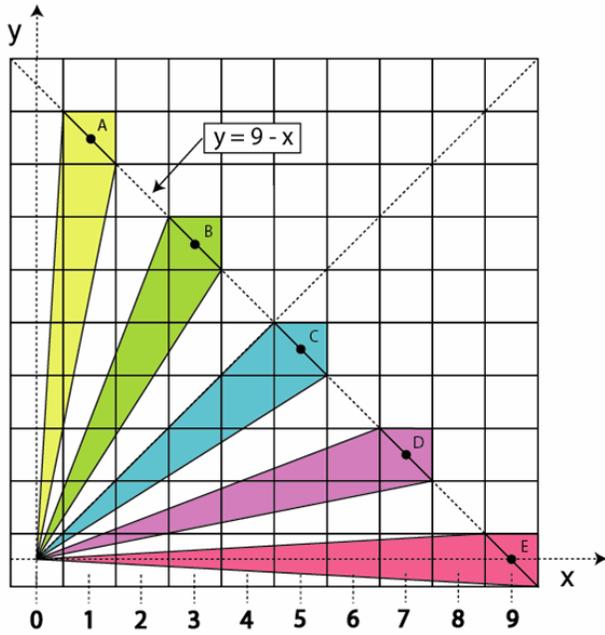


Fig. 1. For a divergent, radially-symmetric surface, such as an inverted cone or a Gaussian hill, the TCA for pixels can be computed analytically. Each *necktie* region shown can be broken into two colored triangles. Triangles below the line $y = m\Delta x - x$ (here $m = 9$) each have base and height given by: $b = \sqrt{2}\Delta x$ and $h = \sqrt{2}m\Delta x/2$, so their area is, $A = m\Delta x^2/2$. Triangles above the line each have an area of $\Delta x^2/2$. This shows that pixels A–D (and those in between) each have $TCA = \Delta x^2(m + 1)/2$. Note, however, that pixel E has $TCA = \Delta x^2(m + \frac{3}{2})/2$.

known as *metric coefficients* determine the local amount of “stretching” associated with the transformation, denoted here by $\rho(u, v)$ and $\sigma(u, v)$. Lengths along curves of constant u and v are obtained by integrating ρ and σ , respectively. The metric coefficients and their product, J , called the *Jacobian* of the transformation are given by

$$\rho(u, v) = \sqrt{x_u^2 + y_u^2} \tag{5}$$

$$\sigma(u, v) = \sqrt{x_v^2 + y_v^2} \tag{6}$$

$$J(u, v) = \rho(u, v) \sigma(u, v). \tag{7}$$

See Peckham (1999) for an application of these concepts to the problem of finding closed-form solutions to a class of nonlinear partial differential equations (PDEs), which includes a PDE for idealized, steady-state landforms.

TABLE I
ORTHOGONAL CURVILINEAR COORDINATE SYSTEMS

	Radial	Parabolic	Elliptic
$x(u, v)$	$u \cos(v)$	$(u^2 - v^2)/2$	$a \cosh(u) \cos(v)$
$y(u, v)$	$u \sin(v)$	uv	$a \sinh(u) \sin(v)$
$\rho(u, v)$	1	$\sqrt{u^2 + v^2}$	$a \sqrt{\sinh^2(u) + \sin^2(v)}$
$\sigma(u, v)$	u	$\rho(u, v)$	$\rho(u, v)$
$u(x, y)$	$\sqrt{x^2 + y^2}$	$-\sqrt{x + r}$	$-\cosh^{-1}[K(x, y)]$
$v(x, y)$	$\tan^{-1}(y/x) + c$	$-y/u(x, y)$	$\sec^{-1}[(a/x)K(x, y)]$

where $r = \sqrt{x^2 + y^2}$, $K(x, y) = \sqrt{G(x/a, y/a)}/2$
and $G(x, y) = 1 + r^2 + \sqrt{1 + 2(r^2 - 2x^2) + r^4}$.

B. General Expressions for TCA and SCA

For any orthogonal curvilinear (OC) coordinate system, the area bounded by two constant- u curves ($u = u_1$ and $u = u_2$) and two constant- v curves ($v = v_1$ and $v = v_2$) is given by the integral

$$A(u_1, u_2, v_1, v_2) = \int_{u_1}^{u_2} \int_{v_1}^{v_2} J(u, v) du dv. \tag{8}$$

The SCA can then be computed (Gallant and Hutchinson, 2011) as

$$a(u_1, u_2, v) = \lim_{\Delta v \rightarrow 0} \frac{A(u_1, u_2, v, v + \Delta v)}{\sigma(u_1, v) \Delta v}. \tag{9}$$

This expression for A can also be used to get approximate values on a rectilinear grid that differ from the exact values by no more than $\Delta x^2/2$. Given closed-form expressions for $u(x, y)$ and $v(x, y)$, the u -value of each grid cell (say u_1) can be computed from the xy coordinates of its center. Similarly, v_1 and v_2 can then be computed from the xy coordinates of two opposite corners of the grid cell. The values u_1, v_1 and v_2 — together with u_2 as the (maximum) u -value of the ridge or peak — can then be inserted into (8) to get the TCA for the given grid cell. SCA can be computed more simply by computing u and v from the xy coordinates of each grid cell’s center and inserting into (9).

III. RADIALLY-SYMMETRIC SURFACES

Any surface with radial symmetry, including a Gaussian hill and an inverted cone, will have its contour lines and streamlines given by a “radial” OC coordinate system (see

Table I, column 1). Contour lines are curves of constant radius, u , and streamlines are curves of constant azimuth angle, v . Elevation is given by $z = H(u)$ for some function, H . Equation (8) then simplifies to

$$A(u_1, v_1; u_2, v_2) = (v_2 - v_1) (u_2^2 - u_1^2) / 2 \quad (10)$$

For a grid cell centered at (x, y) , with sides of length $\Delta y = \Delta x$, and taking $u_2 = 0$, we then have

$$F(x, y) = \tan^{-1}\left(\frac{2y + \Delta x}{2x - \Delta x}\right) - \tan^{-1}\left(\frac{2y - \Delta x}{2x + \Delta x}\right) \quad (11)$$

$$G(x, y) = (x^2 + y^2) / 2 \quad (12)$$

$$A(x, y) = F(x, y) G(x, y). \quad (13)$$

On the diamond-shaped curves where $|x| + |y| = c$, we can eliminate y in $G(x, y)$ and $F(x, y)$. This gives $G(x, y) = (2x^2 - 2c|x| + c^2)/2$, and the first term in a Taylor series of $F(x, y)$ in Δx about 0 gives $F(x, y) \approx c \Delta x / (2x^2 - 2c|x| + c^2)$. As a result, the TCA on these curves is approximately $c \Delta x / 2$, consistent with the exact result obtained in the Introduction. Using (9) to compute the SCA returns us to (4).

IV. SURFACES BASED ON PARABOLIC COORDINATES

The parameters for a *parabolic* OC coordinate system are given in Table I (column 2) where $u \leq 0$ and $v \geq 0$. For any surface with $z = H(u)$, the TCA and SCA can be computed using (8) and (9) as

$$A(u_1, v_1; u_2, v_2) = [(v_2 - v_1) (u_2^3 - u_1^3) + (u_2 - u_1) (v_2^3 - v_1^3)] / 3 \quad (14)$$

$$a(u, v; u_2) = \frac{(u_2^3 - u^3) + 3v^2(u_2 - u)}{3\sqrt{u^2 + v^2}}. \quad (15)$$

For the surface shown in Figure (2), we have $z = u$ with the ridgeline given by the curve $u = u_2 = 0$. SCA can be computed as

$$a(x, y) = \frac{\sqrt{2}}{3} \cdot \frac{[r(2r + x) - x^2]}{\sqrt{r(r + x)}}, \quad (16)$$

where $r = \sqrt{x^2 + y^2}$. Contours of elevation and SCA are shown in Figure (3).

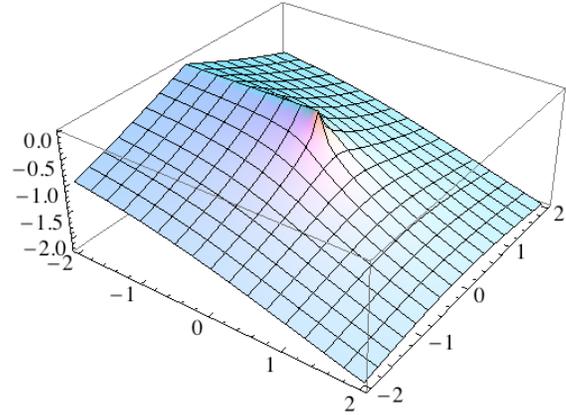


Fig. 2. A divergent surface with contours and streamlines given by a parabolic coordinate system with $z(x, y) = u(x, y)$. The ridgeline is given by $u = 0$ and runs from $x = -\infty$ to $x = 0$ (with $y = 0$).

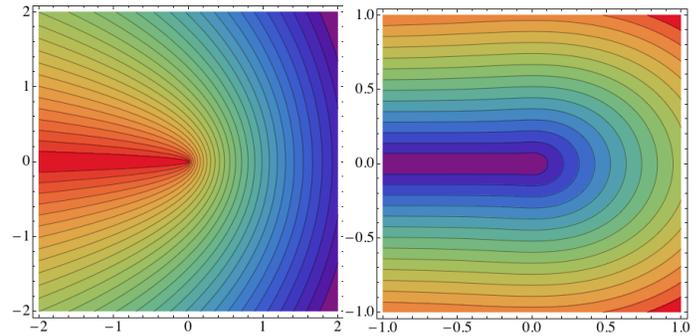


Fig. 3. For the surface in Figure 2: (a) Contours of elevation. Both contour lines and streamlines are given by parabolas. (b) Contours of SCA. SCA is zero on the ridgeline (purple). High values are red and low values are blue.

V. SURFACES BASED ON ELLIPTIC COORDINATES

The parameters for an *elliptic* OC coordinate system are given in Table I (column 3), where $u \leq u_2$ and $v \in [0, 2\pi]$. For any surface with $z = H(u)$, TCA can be computed as

$$A(u_1, v_1; u_2, v_2) = \left(\frac{a}{2}\right)^2 [(u_2 - u_1) F(v_1, v_2) + (v_1 - v_2) G(u_1, u_2)] \quad (17)$$

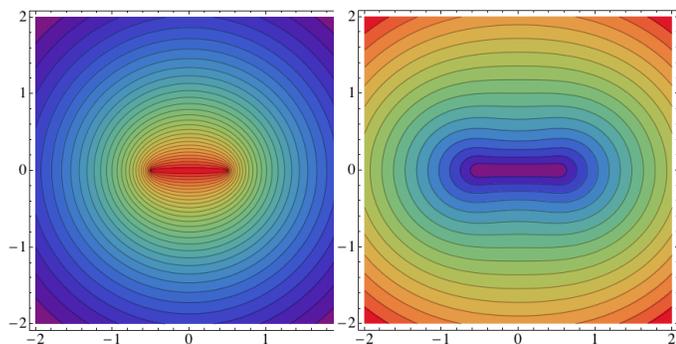


Fig. 4. For the surface based on elliptic coordinates ($z(x, y) = u(x, y)$): (a) Contours of elevation. Contour lines are given by ellipses. Ridgeline is given by $u = 0$, and runs from $x = -a$ to $x = a$ (with $y = 0$). (b) Contours of SCA. SCA is zero on the ridgeline (purple). High values are red and low values are blue.

where

$$F(v_1, v_2) = \sin(2v_1) - \sin(2v_2) \quad (18)$$

$$G(u_1, u_2) = \sinh(2u_1) - \sinh(2u_2). \quad (19)$$

Similarly, the SCA can be computed using (9) as

$$a(u, v; u_2) = \frac{a[B(u_2, v) - B(u, v)]}{4\sqrt{\sin^2(v) + \sinh^2(u)}} \quad (20)$$

$$B(u, v) = \sinh(2u) - 2u \cos(2v). \quad (21)$$

Contours of elevation and SCA are shown in Figure (3) for the case $z = u$, where the ridgeline corresponds to the curve $u = u_2 = 0$.

VI. RESULTS FOR THE MASS FLUX METHOD

Figures (5) and (6) show the SCA as computed using the Mass Flux Method for a radially symmetric surface and for surfaces based on parabolic and elliptic coordinates. Note that the “artifact” that results from the exact TCA calculation is inherited by the SCA figures. The agreement between values computed mathematically and using the Mass Flux Method is quite good. However, work is ongoing to determine how best to remove the artifact. More complete results, including tests of other methods for computing TCA and SCA will be presented in a full paper.

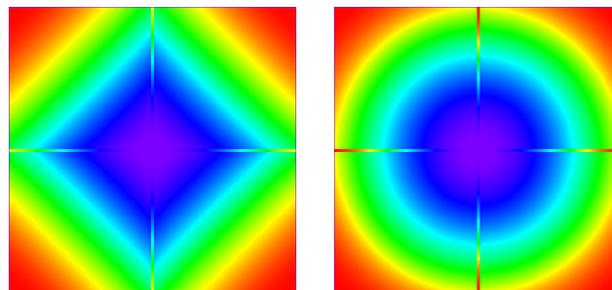


Fig. 5. TCA (a) and SCA (b) for a radially-symmetric surface, as computed with the Mass Flux Method in RiverTools 4.0. High values are red and low values are blue.

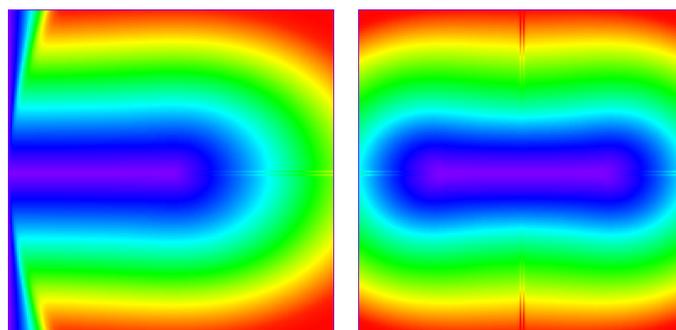


Fig. 6. SCA for surfaces based on parabolic (a) and elliptic (b) coordinates, computed with the Mass Flux Method in RiverTools 4.0. Edge effects are expected in (a). High values are red, low ones are blue.

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